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On Some Minimum-Fuel Problems
I. $\ddot{x}+f(x)=u(t)$

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0. Introduction

Consider a system whose state is described by a vecter $\mathbf{x}$ of an n-dimentional Euclidean space and whose evolution is given by the differential equation

$$\dot{\mathbf{x}}=\mathbf{f}\ (\mathbf{x},\ \mathbf{u},\ t)\ \cdots\cdots\cdots\cdots(1)$$

where $\mathbf{u}$ is an m-dimentional control vecter.

The *minimum-fuel* problem is to search for a control $\mathbf{u}(t)$ which brings the state of the system from $\mathbf{x}_0$ to $\mathbf{x}_f$, and let the fuel $J(\mathbf{u})=\int_0^{t_f}|\mathbf{u}(t)|\,dt$ be minimum, where $\mathbf{x}_0$ is the given initial state and $\mathbf{x}_f$ is the given final state. The minimum fuel problems have been considered by many authors.[1)] Among them we are interested in M. Athans.[2)] He has treated the system which is norm invariant that is $\dot{\mathbf{x}}\cdot\mathbf{x}=O$ for $\mathbf{u}(t)=O$.

The Pontryagin Maximal Principle[3)] states a necessary condition for the optimal control that is an admissible control which gives the minimal fuel $J(\mathbf{u})$. Thus when there is no minimum-fuel, Maximal Principle is not adequate to use. The existence of an admissible control gives rise to the existence of the *greatest lower bound* of the fuels. When there exists the greatest lower bound for fuels and there is no minimum-fuel, is there any theorem like the Maximal Principle ? This is an open question. In the present note this

※) This research was partly supported by the Sangyo Kenkyujo.

1) See Ref. (1), (3), (4), (5), (6), (7)
2) See (5)
3) See (3), (7), (8). Especially in (3) Chapt. 6 by H. Halkin and Chapt. 8 by S.P. Dilberto

4) See (2) pp. 246——289

problem is solved for the special case when the differential equation is $\dot{x} = y$, and $\dot{y} = -f(x) + u(t)$.[4)]

$$I. \quad \ddot{x} + f(x) = u(t)$$

I. 1. Statemens of the main theorem

Let us consider a system whose evolution is given by a differential equation $\dot{x} + f(x) = u(t)$, where $|u(t)| \leq 1$ is a control function. Or equivalently $\dot{x} = y$, $\dot{y} = -f(x) + u(t)$, $|u(t)| \leq 1$. Assume that $f(x)$ is continuous and $f(0) = 0$, and furthermore assume that $f(x) > 0$ on $0 < x < a$ and $f(x)$ <0 on $-b < x < 0$ and $f(a) = f(-b) = 0$.

(If $f(x) > 0$ on $0 < x < \infty$, then let $a = \infty$, and likewise for b)

These conditions imply that $F(x) = \int_0^x f(x)\, dx$ is positive in a nbd (neighborhood) of zero. Put $\mu^2 = \min\{2F(a), 2F(-b)\}$, where F(a) and $F(-b)$ may be ∞ with a or b.

Now the equation of evolution is

$$\begin{cases} \dot{x} = y & \cdots\cdots\cdots\cdots(2) \\ \dot{y} = -f(x) + u(t) & \cdots\cdots\cdots\cdots(3) \end{cases}$$

from (2) and (3) we have $(f(x) - u(t))\,\dot{x} + y\dot{y} = 0$.

Then the trajectories for the equation are

$$y^2 + 2F(x) = \text{const.} \qquad \text{when } u(t) \equiv 0,$$
$$y^2 + 2F(x) + 2x = \text{const.} \qquad \text{when } u(t) \equiv -1,$$
$$y^2 + 2F(x) - 2x = \text{const.} \qquad \text{when } u(t) \equiv 1.$$

When $c^2 \equiv \mu^2$, $y^2 + 2F(x) = c^2$ passes through the points $(l, 0)$, $(-k, 0)$, where l satisfies $2F(l) = c^2$ and $0 < l \leqq a$ and k satisfies $2F(-k) = c^2$ and $-b \leqq -k < 0$, $y^2 + 2F(x) = c^2$ is a loop inclosing the origin.

Let Ω_0 be the open domain inclosed by the loop of the trajectory $y^2 + 2F(x) = \mu^2$.

Lemma 1. *Every point $P(p, q)$ in Ω_0 satisifes $q^2 + 2F(p) < \mu^2$.*

Proof is clear and this Lemma means that there exists one and only one $c \geqq 0$ such that $q_2 + 2F(p) = c^2$.

In order to pave a way of understanding the meaning of the theorem we exhibit an example.

Example 1. $f(x) = x - 3x^2$

$f(x)$ satisfies the conditions reguired, that is $f(x) = x - 3x^2$ is continuous, $f(0) = 0$, $f(x) > 0$ for $0 < x < \frac{1}{3}$, $f(x) < 0$ for $x < 0$, that is $a = \frac{1}{3}$, $-b = -\infty$, and $\mu^2 = \min\left\{2F\left(\frac{1}{3}\right), 2F(-\infty)\right\} = \frac{1}{27}$.

The trajectories are as follows. (see Fig. 1)

$$y^2 + x^2 - 2x^3 = \text{const.} \qquad \text{for } u(t) \equiv 0,$$
$$y^2 + x^2 - 2x^3 + 2x = \text{const.} \qquad \text{for } u(t) \equiv -1,$$
$$y^2 + x^2 - 2x^3 - 2x = \text{const.} \qquad \text{for } u(t) \equiv 1.$$

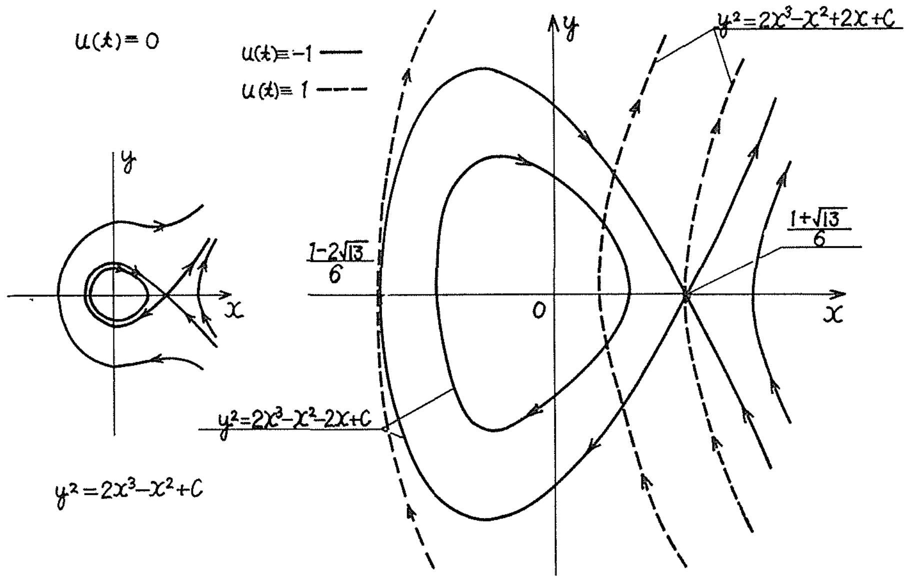

Fig. 1 Trajectories of $\dot{x} = y$, $\dot{y} = -(x - 3x^2) + u(t)$

Now We go on establish the main theorem.

Definition. $r(x, y) = \sqrt{y^2 + 2F(x)}$.

As $F(x)$ is positive in the nbd of 0, $r(x, y)$ is defined in some nbd of the origin, including Ω_0 .

We treat r as if it were a norm. The *directional derivative* of r along the trajectories of the differential equations $\dot{x} = y$, $\dot{y} = -f(x) + u(t)$ is given by

$$\frac{dr}{dt} = \frac{y\dot{y} + f(x)\dot{x}}{\sqrt{y^2 + 2F(x)}} = \frac{y(-f(x) + u(t)) + f(x)y}{r} = \frac{y}{r}u(t) \quad \cdots\cdots\cdots(4)$$

Let $J(u, t) = \int_0^t |u(t)|\,dt$ be the fuel when the particle moves from the time 0 to the time t, Then

$$\frac{dJ}{dt} = |u(t)| \quad \cdots\cdots\cdots(5)$$

By (4) and (5) we have

$$\frac{dr}{dJ} = \frac{u(t)}{|u(t)|} \cdot \frac{y}{r} \quad \cdots\cdots\cdots(6)$$

When the particle starts from the initial position A_0 (x_0, y_0) in Ω_0 and, by a control $u(t)$, reaches the origin at the time t_f, r decreases from $r_0=\sqrt{y_0^2+2F(x_0)}$ to 0 and $J(u, t)$ increases from $J(u,0)=0$ to $J(u, t_f)$.

$\dfrac{dr}{dJ}$ indicates the ratio of the decreasing of r and the increasing of J.

As $\dfrac{dr}{dJ}=\dfrac{u(t)}{|u(t)|}\cdot\dfrac{y}{\sqrt{y^2+2F(x)}}$, $1\geqq\dfrac{dr}{dJ}\geqq-1$ in Ω_0.

Thus we have that for any admissible control $u(t)$, when r decreases from r_0 to 0, J increases from 0 to $r_0+\alpha$, where α is a positive number. This fact is the first part of the following theorem.

Theorem 1. *Let a given system satifies the differential equations* $\dot{x}=y$, $\dot{y}=-f(x)+u(t)$, *where* $u(t)$ *is a control function with* $|u(t)|\leq 1$ *and piecewise continuous. Assume that* $f(x)$ *is continuous and* $f(x)>0$ *in* $(0, a)$, $f(x)<0$ *in* $(-b, 0)$ *and* $f(a)=f(-b)=0$ *(a or b may be* ∞*).*

Let $\mu^2=min\left\{2F(a), 2F(-b)\right\}$, *where* $F(x)=\int_0^x f(x)dx$ *and* $F(a)$ *or* $F(-b)$ *may be* ∞. *Let* Ω_0 *be the domain inclosed by the loop of* $y^2+2F(x)=\mu^2$.

Let A_0 (x_0, y_0) *be a point in* Ω_0, *and let* $u(t)$ *be an admissible control, that is* $x(0)=x_0$, $y(0)=y_0$ *and the equations* $\dot{x}=y$, $\dot{y}=-f(x)+u(t)$ *brings the particle at* A_0 *to the origin at time* t_f. *The required fuel is then* $J(u,t_f)=\int_0^{t_f}|u(t)|dt$. *Then* $J(u, t_f)$ *is not less than* $r_0=\sqrt{y_0^2+2F(x_0)}$.

Converesely for any positive number ε *there exists an admissible control* $u(t)$ *with the fuel less than* $r_0+\varepsilon$.

The first part of the theorem is already proved.

For the proof of the second part of the theorem let us begin with lemmas which are easy to prove. As $f(x)$ is continuous at $x=0$ and $f(0)=0$, for any positive number η we can choose a positive number δ such that $|x|<\delta$ imples $|f(x)|<\eta$. *η and δ are fixed in the next proofs.*

Lemma 1. *There exists a positive number m such that* $m|x|=y^2$ *and* $y+2F(x)=r^2$, $r^2\leq\mu^2$ *imply* $|x|<\delta$.

Proof. As $r^2\leq\mu^2$ the intersection point (x, y) of $m|x|=y^2$ and $y^2=2F(x)=r^2$ is in Ω_0 and so $|y|\leq\mu$, and then $|x|=\dfrac{y^2}{m}\leqq\dfrac{\mu^2}{m}$. We can choose m so that $\dfrac{\mu^2}{m}<\delta$. (See Fig. 2)

Lemma 2. *If* $|x|<\delta$, *then* $\dfrac{F(x)}{|x|}<\eta$.

Proof is clear from the mean value theorem, and the fact $|x|<\delta$ implies $f(\theta x)<\eta$ for $0<\theta<1$.

Lemma 3. *Let us suppose that a particle moves from* $A_1(0, y_1)=(x(t_1), y(t_1))$

where $y_1 > 0$, *to* $A_2(x_2, y_2) = (x(t_2), y(t_2))$ *with evolution given by* $\dot{x} = y$, $\dot{y} = -f(x) - 1$, *and suppose that* A_2 *is on the curve* $mx = y^2$. *Then the following inequalities hold.*

$$(r_1 - r_2)(1 + 2\eta) > t_2 - t_1 > r_1 - r_2$$

$$r_2 < \sqrt{\frac{m + 2\eta}{m + 2 + 2\eta}}\, r_1 ,$$

where $r_1 = y_1 = \sqrt{y_1^2 + 2F(x_1)}$ *and* $r_2 = \sqrt{y_2^2 + 2F(x_2)}$.

Proof. By the mean value theorem we have

$$\frac{r_2 - r_1}{t_2 - t_1} = \frac{dr}{dt}\bigg|_{t = t_1 + \theta(t_2 - t_1)}, \quad 0 < \theta < 1$$

where $\frac{dr}{dt}$ is a directional derivative along the trajectory $y^2 + 2F(x) + 2x = r_1^2$. (or along the evolution given by $\dot{x} = x$, $\dot{y} = -f(x) - 1$).

Along this trajectory

$$\frac{dr}{dt} = \frac{y\dot{y} + f(x)\dot{x}}{r} = -\frac{y}{r} = -\frac{1}{\sqrt{1 + \frac{2F(x)}{y^2}}}$$

as $u(t) = -1$

Let (x, y) be any point on the curve $A_1 A_2$, then $lx = y^2$ for some $l > m$, and so

$$\frac{F(x)}{y^2} = \frac{F(x)}{lx} \text{ (l depends on x)} = \frac{1}{l}\,\frac{1}{x}\int_0^x f(x)dx$$

$$= \frac{1}{l} f(\theta x) < \frac{\eta}{l}$$

Then we have $-\frac{1}{\sqrt{1 + \frac{2\eta}{m}}} > \frac{dr}{dt} > -1$

Then $m \geq 4$ implies that $-1 + \eta > \frac{r_2 - r_1}{t_2 - t_1} > -1$,

or $(r_1 - r_2)(1 + 2\mu) > t_2 - t_1 > r_1 - r_2$.

The proof of the second inequality :

From $mx_2 = y_2^2$ and $y_2^2 + 2F(x_2) + 2x_2 = y_1^2$

$$\frac{y_1^2}{y_2^2 + 2F(x_2)} = 1 + \frac{2x_2}{y_2^2 + 2F(x_2)}$$

$$= 1 + \frac{2}{m + \frac{2F(x_2)}{x_2}}$$

By Lemma 2 $\frac{F(x_2)}{x_2} < \eta$ and the right hand side

$$> 1 + \frac{2}{m + 2\eta} = \frac{m + 2 + 2\eta}{m + 2\eta}$$

This implies that

$$r_2 = \sqrt{y_2^2 + 2F(x_2)} < \sqrt{\frac{m+2\eta}{m+2+2\eta}}\, y_1 = \sqrt{\frac{m+2\eta}{m+2+2\eta}}\, r_1$$

Notice that the inequalities hold when $y_1 < 0$ with replacement of $u(t) = -1$ by $u(t) = 1$.

Lemma 4. *Let* $A(x_1, y_1) = (x(t_1), y(t_1))$ *be a point on the trajectory* $y^2 + 2F(x) + 2x = 0$ *and* $0 > x_1 > -\delta$.

Assume that the particle reaches the origh at time t_2 *with* $u(t) = -1$. *Then* $y_1 < t_2 - t_1 < y_1(1 + 2\eta)$.

Proof.

$$\frac{0 - y_1}{t_2 - t_1} = 1 - f(x) \quad \text{for some} \quad 0 > x > x_1 > -\delta$$

By the hypothesis on δ we have $0 > f(x) > -\eta$.

$$-1 + \eta > \frac{-y_1}{t_2 - t_1} > -1,$$

which imply $y_1(1 + 2\eta) > t_2 - t_1 > y_1$.

The proof of the second part of the theorem : (see Fig. 2)

What is to be proved is : Given an initial point $A_0(x_0, y_0)$ in Ω_0 and a positive number ε we can find an admissible control $u(t)$ whose fuel is less than $r_0 + \varepsilon$.

Without loss of generality we can assume $x_0 > 0$, $y_0 > 0$. Let $A_1(0, y_1)$ be the intersection of the trajectory $y^2 + 2F(x) = r_0^2$ with $y-$ axis. Let t_1 be the time at which the particle reaches A_1 that is $x(t_1) = 0$, $y(t_1) = y_1 = -r_0$.

In other words let $u(t) = 0$ until the particle reaches $y-$ axis. t_1 is clearly finite. Then let $u(t) = 1$ for $t_1 \leq t \leq t_2$ where t_2 is the time at which the particle reaches on $-mx = y^2$ at $A_2(x_2, y_2)$. Then let $u(t) = 0$ for $t_2 \leq t \leq t_3$, where t_3 is the time at which the particle reaches on $y-$ axis at $A_3(0, y_3)$.

Continue these processes and we have points $A_1, A_2, \cdots\cdots\cdots\cdots\cdots, A_{2n+1}$. Let $r_i = \sqrt{y_i^2 + 2F(x_i)}$, then $r_{2i+1} = r_{2i}$ and by Lemma 3

$$r_{2i} < \sqrt{\frac{m+2\eta}{m+2+2\eta}}\, r_{2i-1}. \quad \text{Thus} \quad r_{2n} < \left(\frac{m+2\eta}{m+2+2\eta}\right)^{\frac{n}{2}} r_0.$$

Let $u(t) = 0$ for $t_{2n} < t \leq t_{2n+1}$ and let $A_{2n+1}(x_{2n+1}, y_{2n+1})$ be the intersection of $y^2 + 2F(x) = r_{2n}^2$ and $y^2 + 2F(x) + 2x = 0$, then $x_{2n+1} = \dfrac{-r_{2n}^2}{2}$.

$|x_{2n+1}| < \dfrac{r_{2n}^2}{2}$, which implies $|x_{2n+1}| < \delta$ if $r_{2n}^2 < 2\delta$.

Now the fuel is given by

$J = (t_2 - t_1) + (t_4 - t_3) + \cdots\cdots\cdots\cdots (t_{2n+2} - t_{3n+1})$

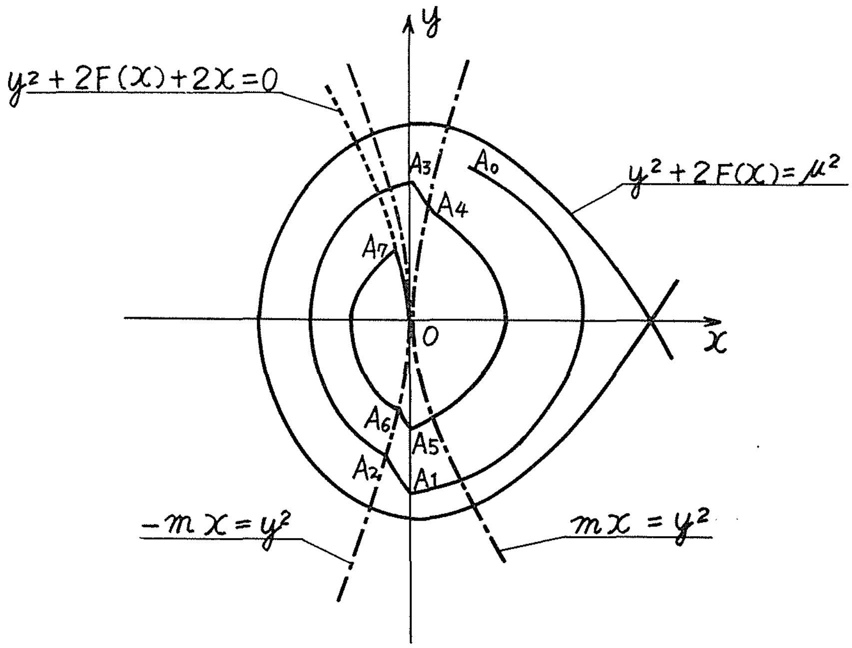

Fig. 2 Trajectory for the control which gives a fuel near the g. l. b of fuels

By Lemma 3 and 4

$$t_{2i} - t_{2i-1} < (r_{2i-1} - r_{2i})\,(1 + 2\eta) \quad i = 1,\ 2, \cdots\cdots, \mathrm{n}-1$$

$$t_{2n+2} - t_{2n+1} < r_{2n+1}\,(1 + 2\eta)$$

$$\begin{aligned} J &< (r_1 - r_2)\,(1 + 2\eta) + (r_3 - r_4)\,(1 + 2\eta) \\ &\qquad + (r_{2n-1} - r_{2n})\,(1 + 2\eta) + r_{2n+1}(1 + 2\eta) \\ &= (1 + 2\eta)\, r_1 \\ &= (1 + 2\eta)\, r_0 \\ &= r_0 + 2\eta\, r_0 \end{aligned} \qquad \left(\begin{array}{l} \text{Noting that} \\ \quad r_0 = r_1, \quad r_2 = r_3, \cdots \\ \quad r_{2n} = r_{2n+1} \end{array} \right)$$

η will be so chosen that $2\eta\, r_0 < \varepsilon$ for the given $\varepsilon > 0$. Thus the proof of the second part of the theorem is end.

I. 2. Examples

Example 1. (Continued) Consider the Example 1 in I. 1. Differential equations $x = y$, $y = -x + 3x^2 + u(t)$, $|u(\mathrm{t})| < 1$.

$$\mu^2 = \frac{1}{27}$$

$\Omega_0 =$ the domain inclosed by the loop of $y^2 + x^2 - 2x^3 = \frac{1}{27}$.

Trajectories :

(i) $y^2+x^2-2x^3=\text{const}$ for $u(t)=0$.

(ii) $y^2+x^2-2x^3-2x=\text{const}$ for $u(t)=1$.

(iii) $y^2+x^2-2x^3+2x=\text{const}$ for $u(t)=-1$.

Let A_0 be $\left(\frac{1}{6}, \frac{1}{9}\right)$, then A_0 is in Ω_0 and

$$r_0=\sqrt{\left(\frac{1}{9}\right)^2+\left(\frac{1}{6}\right)^2-2\left(\frac{1}{6}\right)^3}=\frac{\sqrt{10}}{18}$$

The theorem asserts that to move the particle at A_0 to the origin the greatest lower bound for the fuels is $\frac{\sqrt{10}}{18}$

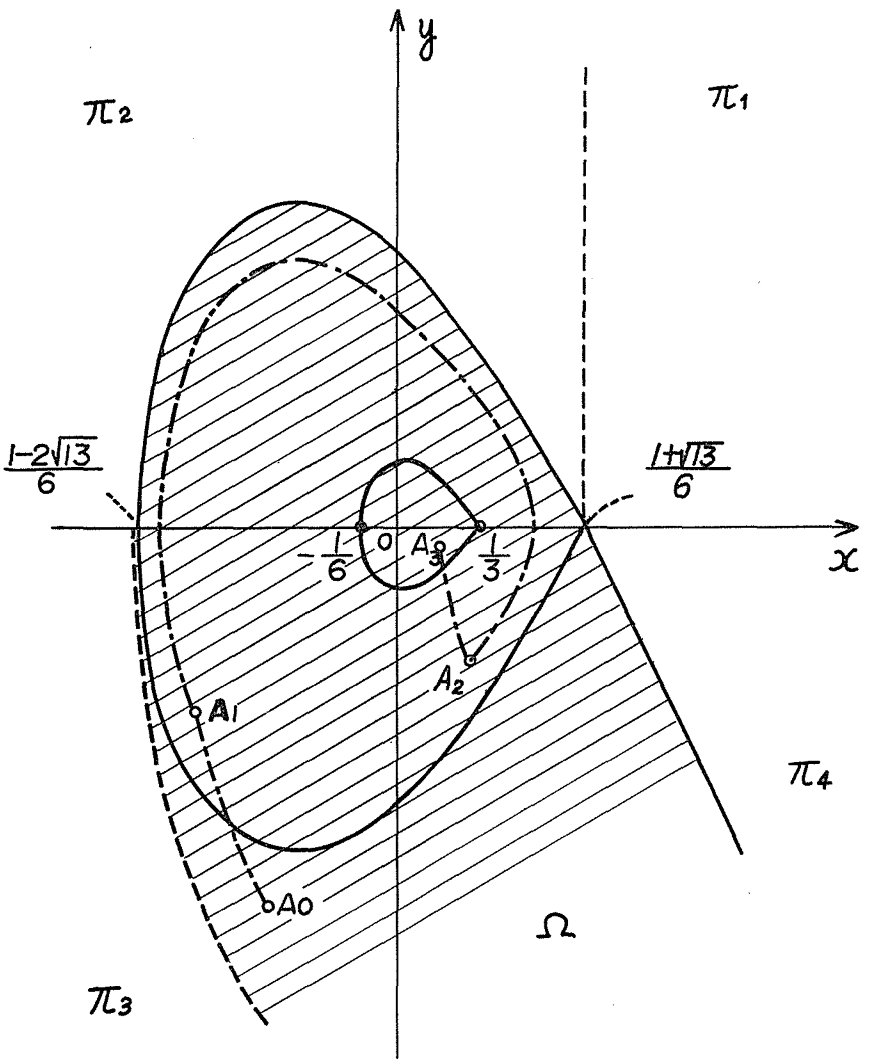

Fig. 3 Controllable region

Let us consider the controllable domain Ω in this example :

$\Omega = \Big\{ (x_0, y_0)$: there exist an admissible control $u(t)$ which brings (x_0, y_0) to the origin $\Big\}$.

If $x \geq \dfrac{1+\sqrt{13}}{6}$, $y>0$, then $\dot{x} = y > 0$ and $\dot{y} = 3x^2 - x - u(t) \geq 0$ when $u(t) \geq -1$. Thus Π_1 in Fig. 3 is not controllable.

In Fig. 3 the particle in Π_2 or Π_4 reaches at Π_1 for every $u(t)$. Likewise the particle in Π_3 reaches at Π_2 for every $u(t)$. Thus $\Pi = \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$ is the uncontrollable domain. The complement of Π is the controllable domain. The initial point A_0 in Fig. 3, for example, by letting successively $u(t) = 1, -1$, moves to A_1, A_2, A_3, and then to the origin.

What is the greatest lower bound of the fuels? This is a question.

Example 2. $\ddot{x} + \sin x = u(t)$, $\quad |u(t)| \leq 1$

Differential equations

$$\dot{x} = y$$
$$\dot{y} = -\sin x + u(t), \qquad |u(t)| \geq 1$$
$$\mu^2 = 4$$

$\Omega_0 =$ the domain inclosed by the loop of $y^2 + 4\sin^2\dfrac{x}{2} = 4$. Trajectories :

$$y^2 + 4\sin^2\frac{x}{2} = \text{const. when } u(t) = 0.$$
$$y^2 + 4\sin^2\frac{x}{2} - 2x = \text{const. when } u(t) = 1.$$
$$y^2 + 4\sin^2\frac{x}{2} + 2x = \text{const. when } u(t) = -1.$$

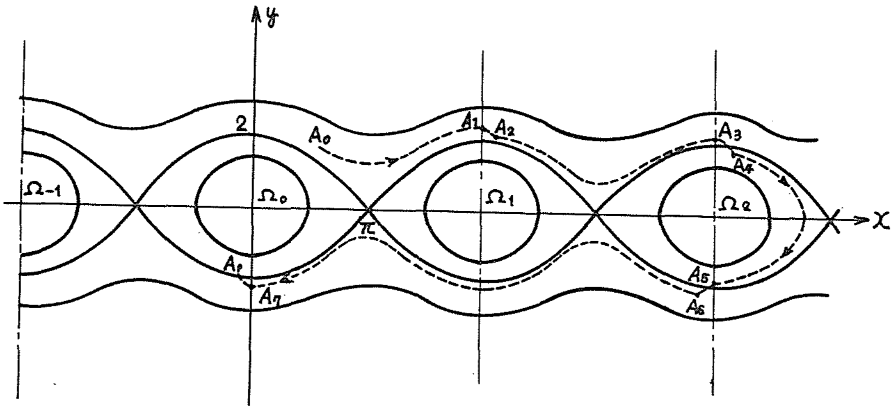

Fig. 4 Trajectory for $\dot{x}=y$, $\dot{y} = -\sin x + u(t)$

In this example the greatest lower bound for the fuels is obtained for every initial point. The domains inclosed by the loops of $y^2+4\sin^2\frac{x}{2}=4$ or $y=\pm 2\cos\frac{x}{2}$ are Ω_0, Ω_1, Ω_2,……, Ω_{-1}, Ω_{-2},……. If an initial point $A_0(x_0, x_0)$ is in Ω_0, then the greatest lower bound for the reguired fuels is $\sqrt{y_0^2+4\sin^2\frac{x_0}{2}}$. If A_0 is in the complement of $\Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \cdots\cdots \cup \Omega_{-1} \cup \Omega_{-2} \cup \cdots\cdots$, then likewise the greatest lower bound for the fuels is $\sqrt{y_0^2+4\sin^2\frac{x^0}{2}}$.

If A_0 is in $\Omega_1 \cup \Omega_2 \cup \cdots\cdots \cup \Omega_{-1} \cup \Omega_{2-} \cup \cdots\cdots$, then the greateatest lower bound for the fuels is $4-\sqrt{y_0^2+4\sin^2\frac{x^0}{2}}$.

This fact is indicated in Fig. 4, that is if A_4 in the figure is the initial point then, to escape from Ω_2 it reqires the fuel not less than $2-\sqrt{y_0^2+4\sin^2\frac{x^0}{2}}$, and from A_6 to A_7 no fuel is needed.

This example can be considered as the problem of stopping the simple pendulum with minimum-fuel. (The name "fuel" is curious in the example. But this name is recognized to mean $J=\int_0^t |u(t)|\,dt$). The restriction $|u(t)|\leqq 1$ does not effect on the value of the minimum-fuel. But the time is effected by the restriction.

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最　小　燃　料　問　題

船山子之助・水　間　卓　一

工 学 部　共通講座

要　　旨

$\mathbf{u}(t)$ を制御ベクトル関数とする状態方程式を $\mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ とする。$t = 0$ における初期状態 $\mathbf{x}_0 = \mathbf{x}(o)$ から時刻 $t = t_f$ で最終状態 $\mathbf{x}_f = \mathbf{x}(t_f)$ に移るとする。このとき，必要燃料（定義）

$$J = \int_0^{t_f} |\mathbf{u}(t)|\, dt$$

を最小とする $\mathbf{u}(t)$ を求めることを最小燃料問題という。

最小燃料問題は他の最適制御の問題と同様，最大値原理その他の方法で研究されているが，いずれも最適制御の存在を仮定して必要条件を求めている。許容制御の存在は燃料の下極限の存在を示しているが，丁度その下極限を与えるような制御は存在しない場合がしばしば起る。

このような場合は理論的にもまた実際上にも最大値原理等は役に立たない。本論文は $x + f(x) = u(t)$ について解答を与えた。$f(x) = k^2 x$ の場合は単振動を最小燃料で静止させることに相当する。